Passage from Einsteinian to Galilean Relativity and Clock Synchrony

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Z. Naturforsch. **46a**, 256–258 (1991); received July 17, 1990

There is a general belief that under small velocity approximation, Special Relativity goes over into Galilean Relativity. Should this be interpreted exclusively in terms of the kinematical symmetry transformations (Lorentz vs. Galilei) a misconception could easily arise that would stem from overlooking the role of conventionality ingredients of Special Relativity Theory. It is observed that the small velocity approximation cannot alter the convention of distant simultaneity. In order to exemplify this point further, the Lorentz transformations are critically compared, under the same approximation, with two other space time transformations, one of which represents an Einstein world with Galilean synchrony whereas the other describes a Galilean world with Einsteinian synchrony.

There seems to be a prevailing belief that Special Relativity (SR) goes over to Galilean Relativity (GR) for relative speeds that are very small compared to the speed of light in vacuum [1–4]. The belief is *typically* expressed in the form that the Lorentz Transformation (LT) goes over to the Galilean Transformation (GT) when β^2 terms, where $\beta = v/c$, are neglected in LT [1, 2]. This assumption, however, is not strictly correct. The aim of the present paper is to demonstrate this statement. We feel that the most straightforward approach is to start from an interesting fallacy posed below.

Consider two events E_1: (x_1, t_1) and E_2: (x_2, t_2) in an inertial frame S. Represented in a Minkowski diagram, the invariant interval between these two events is

$$\begin{aligned}\Delta s^2 &= (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - c^2 (\Delta t)^2 \\ &= (\Delta \bar{x}_1)^2 + (\Delta \bar{x}_2)^2 + (\Delta \bar{x}_3)^2 - c^2 (\Delta \bar{t})^2, \end{aligned} \tag{1}$$

where $\Delta x_i = x_{i2} - x_{i1}$, $\Delta t = t_2 - t_1$ and bars represent the corresponding quantities in another reference frame $\bar{S}$ moving relative to S with the uniform non-zero speed v. If β^2 is neglected and if it were true that LT goes over into GT for $\beta^2 \to 0$, then it should hold that $\Delta \bar{t} = \Delta t$. It follows then from (1) that

$$(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 = (\Delta \bar{x}_1)^2 + (\Delta \bar{x}_2)^2 + (\Delta \bar{x}_3)^2.$$

This leads to a contradiction since, according to GT

$$\Delta \bar{x} = \Delta x - v\,\Delta t, \quad \Delta \bar{y} = \Delta y, \quad \Delta \bar{t} = \Delta t,$$

and clearly, for any two non-simultaneous ($\Delta t \neq 0$) events, $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is not an invariant. The above argument can not be resolved unless one rejects the notation that alone the neglect of β^2 in LT leads to Galilean Relativity. Indeed, if β^2 is neglected in the Lorentz factor, the LT reduces to the Approximate Lorentz Transformation (ALT) [5].

$$\bar{x} = x - v\,t, \quad \bar{t} = t - (v\,x/c^2). \tag{2}$$

Thus, for any pair or events

$$\Delta \bar{x} = \Delta x - v\,\Delta \bar{t}, \quad \Delta \bar{t} = \Delta t - (v/c^2)\,\Delta x. \tag{3}$$

Notice here that for any chosen spatial separation Δx between two events, we can take v sufficiently small, so that Δt becomes very large compared to $(v/c^2)\,\Delta x$ and hence the latter may be neglected implying $\Delta \bar{t} = \Delta t$. On the other hand, the approximation $v^2/c^2 \ll 1$ is certainly independent of the space time separation of two arbitrary and independent events. In fact, for any preassigned small value of v one is free to consider a pair of sufficiently distant events so that one cannot ignore the $(v/c^2)\,\Delta x$ term in (3). Therefore absolute nature of distant simultaneity ($\Delta \bar{t} = \Delta t$) can never be retrieved. That is, simultaneity is still relative. This is not surprising since we should realize that the relative character of distant simultaneity is the result of a synchronization convention [6–13]. A convention once chosen a priori is unlikely to change into a different

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one merely due to an approximative assumption on the relative velocity alone.

Let us recall that the standard Einstein synchronization procedure requires spatially distant clocks to be so adjusted that in any given inertial frame the to and fro speeds of light appear to be the same and equal to the round trip speed of light [6–12]. In this context it is now worthwhile to examine, in some detail the nature of ALT(2) for *all* v.

The velocity addition laws can be obtained from (2) as

$$\overline{W}_x=(W_x-v)/[1-(v W_x/c^2)],$$
$$\overline{W}_y=W_y/[1-(v W_x/c^2)].$$

As expected, W_y does not transform as in SR. Now, if a light pulse is sent back and forth along the x-direction alone, the to and fro speed of light in $\bar{S}$, parallel to the direction of motion, is given by

$$C_{\parallel}=c\,. \tag{4}$$

If, on the other hand, a light pulse is sent back and forth in S in such a direction that the signals travel back and forth only in the y-direction in $\bar{S}$, one obtains, using the fact that $W_x^2+W_y^2=C^2$ in S, for the speed of light in $\bar{S}$, perpendicular to the direction of motion, the value

$$C_{\perp}=\frac{c}{(1-\beta^2)^{1/2}}\,. \tag{5}$$

These results, i.e. (4) and (5), certainly do not agree with the corresponding classical results unless $v=0$ strictly (NB, the classical result $C_{\parallel}=c(1\pm\beta)$ differs from (4) in the first order of β!). Furthermore, from (4) and (5) we see that the to and fro speeds are individually equal both in the longitudinal direction and in the transverse direction. In fact, it can be shown that the same conclusion holds also for any arbitrary direction in $\bar{S}$. This is precisely the standard synchronization convention. Thus Einsteinian synchrony inherent in LT is preserved (even under the approximation $\beta^2 \ll 1$). This is exactly in accordance with our earlier assertion.

However, one may still suspect whether the transformation (2) represents a Galilean world in essence, save the synchronization convention. In order to decide this, one must compare synchrony independent quantities obtained from (2) with those obtained from the usual Galilean transformations. One such quantity is the round trip speed of any signal. In fact, two sets of transformations may appear structurally very different depending on the choice of synchrony, but when synchrony independent quantities are compared one might discover that they are essentially the same. In that case we say that these two transformations represent the same kinematical "World". From the Galilean transformation, it follows that the two-way average speed of light in the direction parallel and perpendicular to the direction of relative motion are given, respectively, by

$$\overleftrightarrow{C}_{\parallel}=c(1-\beta^2) \tag{6}$$

and

$$\overleftrightarrow{C}_{\perp}=c(1-\beta^2)^{1/2}, \tag{7}$$

whereas we see from (4) and (5) that they are given by

$$\overleftrightarrow{C}_{\parallel}=c, \tag{8}$$
$$\overleftrightarrow{C}_{\perp}=c/(1-\beta^2)^{1/2}. \tag{9}$$

Thus, (2) for all v in general, does not represent a Galilean World (GW). Of course one may choose $\beta^2 \ll 1$ again in (6), (7), and (9), and it becomes clear that (2) represents a GW approximately. But then there is a subtle point that must be carefully noted. The resulting GW is not a GW *in totality* but it is limited by the very approximation. To exemplify this point, consider the Tangherlini Transformation (TT), which represents an Einstein World (EW) with absolute (Galilean) synchrony [14]:

$$\bar{x}=(x-vt)/(1-\beta^2)^{1/2},\quad \bar{t}=t(1-\beta^2)^{1/2}. \tag{10}$$

Note here that if $\beta^2 \ll 1$, the resulting transformations represent a GT in totality. Obviously, this fact is absent in (2).

Thus we have demonstrated that the LT does not lead under the small velocity approximation to Galilean (absolute) synchrony. As a result, the Galilean transformation law for *one way* velocities could not be obtained unless $v=0$ strictly. However, (2) represents a GW only for small velocities but not for the entire velocity range, in contrast to the Tangherlini case just mentioned above.

Finally, one may raise the question whether it is at all possible to construct a transformation which represents a GW in totality having Einstein synchrony. Indeed, one may verify that the transformation (ZST)

$$\bar{x}=x-vt,\quad \bar{t}=\frac{t-(vx/c^2)}{1-\beta^2}, \tag{11}$$

due to Zahar and Sjödin [10, 12, 15], satisfies the above characteristics which are just complementary to those of the Tangherlini Transformation. It is evident that the ZST transformation reduces to ALT from (2) if the β^2 term is neglected. Note that here again the Poincaré-Einstein synchrony is preserved.

Thus we see that LT under the small velocity approximation does not go over to GT but instead, it becomes, as it should be equivalent to ZST from (11) under the same approximation. In contrast, TT from (10) directly goes over to GT. Therefore, in order to fully comprehend the passage of SR to GR one should examine LT vis-a-vis ZST and GT vis-a-vis TT in the context of the small speed approximation.

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